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AN ALTERNATE DERIVATION AND

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AN ALTERNATE DERIVATION AND
INTERPRETATION OF THE DRIFT-MINIMUM PRINCIPLE

NASA Contract NASw-563

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FOREWORD

This document is one of sixteen sections that comprise the final report prepared by the Minneapolis-Honeywell Regulator Company for the National Aeronautics and Space Administration under contract NASw-563. The report is issued in the following sixteen sections to facilitate updating as progress warrants:

1541-TR 1	Summary
1541-TR 2	Control of Plants Whose Representation Contains Derivatives of the Control Variable
1541-TR 3	Modes of Finite Response Time Control
1541-TR 4	A Sufficient Condition in Optimal Control
1541-TR 5	Time Optimal Control of Linear Recurrence Systems
1541-TR 6	Time-Optimal Bounded Phase Coordinate Control of Linear Recurrence Systems
1541-TR 7	Penalty Functions and Bounded Phase Coordinate Control
1541-TR 8	Linear Programming and Bounded Phase Coordinate Control
1541-TR 9	Time Optimal Control with Amplitude and Rate Limited Controls
1541-TR 10	A Concise Formulation of a Bounded Phase Coordinate Control Problem as a Problem in the Calculus of Variations
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1541-TR 12	State Determination for a Flexible Vehicle Without a Mode Shape Requirement
1541-TR 13	An Application of the Quadratic Penalty Function Criterion to the Determination of a Linear Control for a Flexible Vehicle
1541-TR 14	Minimum Disturbance Effects Control of Linear Systems with Linear Controllers
1541-TR 15	An Alternate Derivation and Interpretation of the Drift-Minimum Principle
1541-TR 16	A Minimax Control for a Plant Subjected to a Known Load Disturbance

Section 1 (1541-TR 1) provides the motivation for the study efforts and objectively discusses the significance of the results obtained. The results of inconclusive and/or unsuccessful investigations are presented. Linear programming is reviewed in detail adequate for sections 6, 8, and 16.

It is shown in section 2 that the purely formal procedure for synthesizing an optimum bang-bang controller for a plant whose representation contains derivatives of the control variable yields a correct result.

In section 3 it is shown that the problem of controlling m components $(1 < m \leq n)$, of the state vector for an n-th order linear constant coefficient plant, to zero in finite time can be reformulated as a problem of controlling a single component.

Section 4 shows Pontriagin's Maximum Principle is often a sufficient condition for ptimal control of linear plants.

Section 5 develops an algorithm for computing the time optimal control functions for plants represented by linear recurrence equations. Steering may be to convex target sets defined by quadratic forms.

In section 6 it is shown that linear inequality phase constraints can be transformed into similar constraints on the control variables. Methods for finding controls are discussed.

Existence of and approximations to optimal bounded phase coordinate controls by use of penalty functions are discussed in section 7.

In section 8 a maximum principle is proven for time-optimal control with bounded phase constraints. An existence theorem is proven. The problem solution is reduced to linear programming.

A backing-out-of-the-origin procedure for obtaining trajectories for time-optimal control with amplitude and rate limited control variables is presented in section 9.

Section 10 presents a reformulation of a time-optimal bounded phase coordinate problem into a standard calculus of variations problem.

A mathematical method for assessing the approximation of a system by a lower order representation is presented in section 11.

Section 12 presents a method for determination of the state of a flexible vehicle that does not require mode shape information.

The quadratic penalty function criterion is applied in section 13 to develop a linear control law for a flexible rocket booster.

In section 14 a method for feedback control synthesis for minimum load disturbance effects is derived. Examples are presented.

Section 15 shows that a linear fixed gain controller for a linear constant coefficient plant may yield a certain type of invariance to disturbances. Conditions for obtaining such invariance are derived using the concept of complete controllability. The drift minimum condition is obtained as a specific example.

In section 16 linear programming is used to determine a control function that minimizes the effects of a known load disturbance.

TABLE OF CONTENTS

AN ALTERNATE DERIVATION AND INTERPRETATION OF THE DRIFT-MINIMUM PRINCIPLE*

By C. A. Harvey‡

ABSTRACT

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The design of a control system for a plant with disturbance inputs is considered. The motion of the plant is assumed to be described by n linear, first-order, constant-coefficient differential equations forced with a scalar control variable and a scalar disturbance. A linear, fixed-gain controller is assumed. In some cases it is possible to choose the gains in such a manner that a certain type of invariance to disturbances is obtained for the resulting controlled system. Conditions for obtaining such invariance are derived using the concept of complete controllability. The Drift-Minimum condition is obtained as a specific example.
Author

INTRODUCTION

The system considered can be written in the form of the vector differential equation

$$\dot{x} = Ax + bu(x,g) + eg \tag{1}$$

where x is an n-vector representing the state of the system, dot represents differentiation with respect to the independent variable t, A is a constant nxn matrix, b and e are constant n-vectors, u(x,g) is a scalar feedback control, and g is a scalar disturbance considered to be a function of t only. The controller, u(x,g) is

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assumed to be given by

$$u(x,g) = \sum_{i=1}^{n} k_i x_i + k_{n+1} g \tag{2}$$

for some set of n+1 constants, k_i, $i = 1,2,\ldots,n+1$. Denoting by k the vector with components k_i, $i = 1,2,\ldots,n+1$, it is possible to substitute for $u(x,g)$ in (1) the right-hand side of (2) to obtain

$$\dot{x} = A(k)x + e(k)g \tag{3}$$

where the ij^{th} element of $A(k)$ is $a_{ij} + b_i k_j$ and the i^{th} element of $e(k)$ is $e_i + b_i k_{n+1}$.

For certain choices of the vector k, there may be some linear combination of the components of x that is completely insensitive to the disturbance. Such controllers are called invariant controllers (reference 3). The concept of complete controllability (reference 2) is related to the sensitivity of a system to a control variable. It is shown in the following paragraph that invariance to the disturbance is obtained if and only if the system (3) is not completely controllable when g is considered as the control variable. As an explicit example the Drift-Minimum condition (reference 1) is derived as a type of invariance condition. That is, the Drift-Minimum condition insures a certain insensitivity to disturbances.

COMPLETE CONTROLLABILITY AND INVARIANCE

For equation (3) the concept of complete controllability can be considered with respect to the disturbance $g(t)$ if $g(t)$ is thought of as a control. The condition for the system represented by (3) to be completely controllable with respect to $g(t)$ is that the matrix with columns $e(k)$, $A(k)e(k),\ldots,[A(k)]^{n-1}e(k)$ be non-singular, i.e.

$$\det|e(k), A(k)e(k), \ldots, [A(k)]^{n-1} e(k)| \neq 0. \tag{4}$$

If (3) is completely controllable with respect to g(t) then, for arbitrary n-vectors x^o and x^1 and arbitrary real numbers t_o and t_1, there exists a g(t) depending on x^o, x^1, t_o, and t_1 defined on $[t_o, t_1]$ such that the response corresponding to g(t) and the initial condition $x(t_o) = x^o$ satisfies $x(t_1) = x^1$. This means, loosely, that no linear combination of components of x(t) is insensitive (invariant) to disturbances g(t). Conversely, if the system (3) is not completely controllable with respect to g(t), i.e., (4) does not hold, then some linear combination of the components of x(t) is insensitive (invariant) to all disturbances g(t).

The theory of invariance also deals with the insensitivity of a system to disturbances. The system represented by (3) would be completely invariant if e(k) = 0. Then, clearly the system is invariant with respect to all disturbances g(t). In general this is impossible to attain since e(k) has n components and the only parameter available to the control designer which appears in e(k) is k_{n+1}. Selective invariance is obtained when some linear combinations of the components of x(t) are invariant relative to g(t). It is clear that invariance can only be attained when (4) is violated. Thus to find invariant controllers it suffices to find solutions of the following equation

$$\det|e(k), A(k)e(k), \ldots, [A(k)]^{n-1} e(k)| = 0 \tag{5}$$

In the general case, equation (5) involves all the gains k_i, $i = 1,2,\ldots,n+1$. Hence, in general, equation (5) may be satisfied with some freedom of choice of certain of the k_i to achieve adequate control of the total system. However, in practice this may not be the case.

One important point to be noted is that, if equation (5) is satisfied, there is a linear combination of the components of $x(t)$ which is invariant. This linear combination can be found, but determining its physical meaning may be difficult.

EXAMPLE

The example to be considered is concerned with the control of a rigid vehicle. The Drift-Minimum control mode will be obtained as a selective invariant controller for a space vehicle. The notation and equations of motion are from reference 1. Also a brief derivation of the drift-minimum condition is given based upon reference 1.

List of Symbols

F	Thrust force
X	Axial air force
N	Air force perpendicular to long axis of vehicle
R	Control force perpendicular to long axis of missile
m	Mass of vehicle
c_1	Specific aerodynamic restoring torque
c_2	Specific control torque
v	Magnitude of standard velocity of vehicle
w	Wind velocity magnitude
$\ddot{z}$	Linear acceleration of center of gravity of vehicle perpendicular to standard path

$\dot{z}$ Velocity of center of gravity of vehicle perpendicular to standard path

$\ddot{\tau}$ Linear acceleration of center of gravity of vehicle perpendicular to long axis of vehicle

a Local linear acceleration at a vehicle station perpendicular to long axis of vehicle.

α Angle of attack

β Swivel motor deflection or vane deflection

ϕ Attitude angle

α_w Wind angle, between flow and standard path

a_o Attitude displacement gain

a_1 Attitude rate gain

a_2 Attitude acceleration gain

b_o Angle of attack gain

g_2 Local lateral acceleration gain

C_G Location of center of gravity

C_M Location of accelerometer

others explained at place of occurrence

Superscripts

dot Differentiation with respect to time

prime Differentiation with respect to angle

The idealized, linearized equations of motion are:

Lateral Path Motion:

$$\ddot{z} = \frac{F-X}{m} \phi + \frac{N'}{m} \alpha + \frac{R'}{m} \beta \tag{6}$$

Angular Motion:

$$\ddot{\phi} = -c_1 \alpha - c_2 \beta \tag{7}$$

Angular Relationship:

$$\alpha - \alpha_w = \phi - \frac{\dot{z}}{v} \; ; \; \alpha_w \equiv \frac{w}{v} \tag{8}$$

The control equation has the form

$$\beta = a_o \, \phi + a_1 \, \dot{\phi} + \underbrace{b_o \, \alpha + g_2 \, a_M}_{\text{In applications, the two terms are mutually exclusive.}} \tag{9}$$

In the case when an accelerometer is used the control equation can be modified by use of the following relations.

$$a_M = \ddot{\tau} + (C_M - C_G) \, \ddot{\phi} \tag{10}$$

$$a_2 = g_2 \, (C_M - C_G) \tag{11}$$

$$\ddot{\tau} = \frac{N'}{m} \alpha + \frac{R'}{m} \beta \tag{12}$$

Then equation (9) can be replaced by

$$(1 - g_2 \frac{R'}{m}) \, \beta = \alpha_o \phi + a_1 \dot{\phi} + a_2 \ddot{\phi} + (g_2 \frac{N'}{m} + b_o) \alpha \tag{13}$$

To simplify notation let $\gamma_1 = N'/m$, $\gamma_2 = F\text{-}X/m$ and $\gamma_3 = R'/m$. Using equations (8) and (13), α and β may be eliminated from equations (6) and (7). The result is:

$$\begin{aligned} &(1 + a_2 c_2 - g_2 \gamma_3) \ddot{\phi} + a_1 c_2 \dot{\phi} + [c_1 + c_2 (a_o + b_o) + g_2 (c_2 \gamma_1 - c_1 \gamma_3)] \phi \\ &= [c_1 + c_2 b_o + g_2 (c_2 \gamma_1 - c_1 \gamma_3)] \, (\dot{z} - w)/v \end{aligned} \tag{14}$$

$$\begin{aligned} &(1 - g_2 \gamma_3) \ddot{z} + (\gamma_1 + b_o \gamma_3)(\dot{z} - w)/v = a_2 \gamma_3 \ddot{\phi} + a_1 \gamma_3 \dot{\phi} \\ &+ [(a_o + b_o - g_2 \gamma_2) \gamma_3 + \gamma_1 + \gamma_2] \phi \end{aligned} \tag{15}$$

The term $(\dot{z} - w)/v$ can be eliminated from equations (14) and (15) resulting in the equation

$$[c_1+c_2b_o + g_2(c_2\gamma_1-c_1\gamma_3)]\ddot{z} = - [(1+a_2c_2)\gamma_1 + (b_o+a_2c_1)\gamma_3]\ddot{\phi}$$
$$-a_1(c_2\gamma_1-c_1\gamma_3)\dot{\phi} + [(c_2\gamma_1-c_1\gamma_3)(g_2\gamma_2-a_o)+(c_1+c_2b_o)\gamma_2]\phi \tag{16}$$

The quasi-steady state solutions for α, β and ϕ are defined as the solutions which result if first and higher derivatives of these variables are assumed to be zero. The constraint that there shall be no lateral acceleration ($\ddot{z}$) on the center of gravity of the vehicle for any steady state, ϕ, has been designated as the "Drift-Minimum-Principle . Since in transient motion, where $\dot{\phi}$ and $\ddot{\phi}$ are of finite values, also transient non-zero values for $\ddot{z}$ are to be expected, the claim for zero-drift cannot be made. From (16) it is clear that this condition may be written down explicitly as

$$(c_2\gamma_1-c_1\gamma_3)(g_2\gamma_2-a_o) + \gamma_2(c_1+c_2b_o)=0 \tag{17}$$

Now the equations of motion will be manipulated slightly and complete controllability with respect to α_w will be investigated. The case of $g_2 = 0$, that is when no accelerometer is present, will be discussed first. Subsequently the case when $b_o = 0$ will be discussed.

Let $x_1 = \phi$, $x_2 = \dot{\phi}$ and $x_3 = \dot{z}$. When $g_2 = 0$, equations (14) and (15) can then be written as the vector differential equation:

$$\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ d_1-c_2a_o & -c_2a_1 & -d_1/v \\ d_2+\gamma_2+\gamma_3a_o & \gamma_3a_1 & -d_2/v \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} + \begin{vmatrix} 0 \\ d_1 \\ d_2 \end{vmatrix} \alpha_w \tag{18}$$

where $d_1 = - c_1 - c_2b_o$ and $d_2 = \gamma_1 + \gamma_3b_o$. This is equivalent to $\dot{x} = Bx + d\alpha_w$ with

$$B = \begin{vmatrix} 0 & 1 & 0 \\ d_1 - c_2a_o & -c_2a_1 & -d_1/v \\ d_2 + \gamma_2 + \gamma_3a_o & \gamma_3a_1 & -d_2/v \end{vmatrix} \quad \text{and } d = \begin{vmatrix} 0 \\ d_1 \\ d_2 \end{vmatrix}$$

Before proceeding with the analysis of this system, the equations will be written in the form discussed in the introduction. Equations (6) and (7) can be written using (8) in the form:

$$x = Ax + bu + eg$$

where $u = \beta$, $g = \alpha_w$, $a_{11} = a_{13} = a_{22} = a_{23} = 0$, $a_{12} = 1$, $a_{21} = - c_1$, $a_{23} = - c_1/v$, $a_{31} = \gamma_1 + \gamma_2$, $a_{33} = -\gamma_1/v$, $b_1 = 0$, $b_2 = -c_2$, $b_3 = \gamma_3$, $e_1 = 0$, $e_2 = -c_1$, $e_3 = \gamma_1$. Equation (9) can be written (with $g_2 = 0$ and using (8)) as

$$u = k_1x_1 + k_2x_2 + k_3x_3 + k_4\, g$$

where $k_1 = a_0 + b_0$, $k_2 = a_1$, $k_3 = - b_0/v$ and $k_4 = b_0$. Then equation (18) is of the form (3) with $B = A(k)$ and $d = e(k)$. It should be noted in this case that the four gains k_1, k_2, k_3, k_4 are not independent since they depend on the three gains a_0, a_1 and b_0.

Now

$$Bd = \begin{bmatrix} d_1 \\ -d_1(c_2a_1 + \frac{d_2}{v}) \\ \gamma_3a_1d_1 - \frac{d_2^2}{v} \end{bmatrix} \text{ and } B^2d = \begin{bmatrix} -d_1(c_2a_1 + \frac{d_2}{v}) \\ d_1(d_1 - c_2a_o + c_2^2a_1^2 + \frac{c_2a_1}{v} - \frac{\gamma_3a_1d_1}{v} + \frac{d_2^2}{v^2}) \\ d_1(d_2 + \gamma_2 + \gamma_3a_o - c_2\gamma_3a_1^2 - \frac{2\gamma_3a_1d_2}{v}) + \frac{d_2^3}{v^2} \end{bmatrix}$$

The determinant of the matrix with columns d, Bd and B^2d can be easily computed and after simplification may be written:

$$\det|d,\ Bd,\ B^2d| = -d_1^2\ (\gamma_2\ d_1 + \gamma_3\ a_o\ d_1 + c_2\ a_o\ d_2) \tag{19}$$

From this equation it is clear that there are two conditions under which the system fails to be completely controllable with respect to the wind, namely: $d_1 = 0$ or

$$\gamma_2\ d_1 + a_o\ (\gamma_3\ d_1 + c_2\ d_2) = 0 \tag{20}$$

It can be readily verified, using the definitions of d_1, d_2, γ_2 and γ_3, that equation (20) is the same as equation (17) if $g_2 = 0$. That is, equation (20) is the drift-minimum condition in the case when $g_2 = 0$. When this condition is satisfied the system is not completely controllable, and hence there is some variable (y), a linear combination of x_1, x_2, and x_3, that is invariant with respect to the wind. This variable, y, is explicitly

$$y = (d_1)^{-1}\ [a_1(c_2\gamma_1 - c_1\gamma_3)x_1 + d_2x_2 - d_1x_3] \tag{21}$$

and $\dot{y} = 0$. Thus the drift-minimum condition is in fact the zero y condition.

The other condition for which the system fails to be completely controllable is $d_1 = 0$. If this condition holds it is clear from equation (18) that $x_1 = \phi$ and $x_2 = \dot{\phi}$ are invariant with respect to α_w. In this case

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -\ c_2a_ox_1 - c_2a_1x_2$$

and if x_1 and x_2 are initially zero, $\dot{x}_3 = \dfrac{\gamma_1c_2-\gamma_3c_1}{c_2V}\ (-x_3 + w)$. Here the behavior of x_3 with respect to w is independent of the choice of the free gains a_o and a_1 since their values only influence the invariant portion of the system.

For the case when the accelerometer is used, equations (14) and (15) can be written (with x_1, x_2, x_3, γ_1, γ_2, γ_3 as defined above) as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ f_1 - a_o c_2 \gamma_4 & -a_1 c_2 \gamma_4 & -\frac{f_1}{v} \\ f_2 + \gamma_2 + a_o \gamma_3 \gamma_4 & a_1 \gamma_3 \gamma_4 & -\frac{f_2}{v} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ f_1 \\ f_2 \end{bmatrix} \alpha_w \tag{22}$$

where $\gamma_4 = (1 + a_2 c_2 - g_2 \gamma_3)^{-1}$, $f_1 = -\gamma_4 [c_1 + g_2(c_2\gamma_1 - c_1\gamma_3)]$ and $f_2 = \gamma_4[\gamma_1 + a_2\ (c_2\gamma_1 - c_1\gamma_3)]$. Equation (22) can be written as $\dot{x} = Cx + f\alpha_w$ where the meanings of C and f are clear. Then the determinant of the matrix with columns f, Cf, and C^2f can be computed. The result is

$$\det \left| f,\ Cf,\ C^2 f \right| = -\ f_1^2\ [\gamma_2 f_1 + a_o \gamma_3 \gamma_4 f_1 + a_o c_2 \gamma_4 f_2] \tag{23}$$

From this equation it is apparent that the two conditions for which the system fails to be completely controllable with respect to the wind are $f_1 = 0$ or

$$\gamma_2 f_1 + a_o \gamma_3 \gamma_4 f_1 + a_o c_2 \gamma_4 f_2 = 0 \tag{24}$$

It can be readily verified that equation (24) is the same as equation (17) with $b_o = 0$. Thus in this case equation (24) is the drift-minimum condition. At this point statements similar to those made in the first case could be made.

CONCLUSIONS

For the type of system considered an explicit condition for selective invariant controllers is obtained using the concept of complete controllability. The Drift-Minimum controller is obtained as a selective invariant controller in the example considered.

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